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Rates of Change of Eigenvalues and Eigenvectors in Damped Dynamic System

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Rates of change of eigenvalues and eigenvectors of a damped linear discrete dynamic system with respect to the system parameters are presented. A nonproportional viscous damping model is assumed. Because of the nonproportional nature of the damping, the mode shapes and natural frequencies become complex, and as a consequence the sensitivities of eigenvalues and eigenvectors are also complex. The results are presented in terms of the complex modes and frequencies of the second-order system, and the use of rather undesirable state-space representation is avoided. The usefulness of the derived expressions is demonstrated by considering an example of a nonproportionally damped two-degree-of-freedom system.

I. Introduction

CHANGES of the eigenvalues and eigenvectors of a linear vibrating system caused by changes in system parameters are of wide practical interest. Motivation for this kind of study arises, on one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualize the changes in the dynamic response with respect to system parameters. Besides, this kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods. Rates of change of eigenvalues and eigenvectors are useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities because of this slight alteration. Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems. Possibly, the earliest work on the sensitivity of the eigenvalues was carried out by Rayleigh.[1] In his classic monograph he derived the changes in natural frequencies caused by small changes in system parameters. Fox and Kapoor[2] have given exact expressions for rates of change of eigenvalues and eigenvectors with respect to any design variables. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure in its current state and have been used extensively in a wide range of application areas of structural dynamics. Nelson[3] proposed an efficient method to calculate eigenvector derivative, which requires only the eigenvalue and eigenvector under consideration. A comprehensive review of research on this kind of sensitivity analysis can be obtained in Adelman and Haftka.[4]

The analytical methods just mentioned are based on the undamped free vibration of the system. For damped systems, unless the damping matrix of the structure is proportional to the inertia and/or stiffness matrices (proportional damping) or can be represented in the series form derived by Caughey,[5] the mode shapes of the system will not coincide with the undamped mode shapes. In the presence of general nonproportional viscous damping, the equations of motion in the modal coordinates will be coupled through the off-diagonal terms of the modal damping matrix, and the mode shapes and natural frequencies of the structure will in general be complex. The solution procedures for such nonproportionally damped systems follow mainly two routes: the state-space method and approximate methods in N space. The state-space method,[6] although exact in nature, requires significant numerical effort for obtaining the eigensolutions as the size of the problem doubles. Moreover, this method also lacks some of the intuitive simplicity of traditional modal analysis. For these reasons there has been considerable research effort to analyze nonproportionally damped structures in N space. Most of these methods either seek an optimal decoupling of the equations of motion or simply neglect the off-diagonal terms of the modal damping matrix. Following such methodologies, the mode shapes of the structure will still be real. The accuracy of these methods, other than the light damping assumption, depends on various factors, for example, frequency separation between the modes, driving frequency, etc. (see Refs. 7 and 8 and the references therein for discussions on these topics). A convenient way to avoid the problems that arise due to the use of real normal modes is to incorporate complex modes in the analysis. Apart from the mathematical consistency, conducting experimental modal analysis one also often identifies complex modes: as Sestieri and Ibrahim[9] have put it, "... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!" But surprisingly in most of the current application areas of structural dynamics that utilize the eigensolution derivatives, e.g., modal updating, damage detection, design optimization, and stochastic finite element methods, they do not use complex modes in the analysis but rely on the real undamped modes only. This is partly because of the problem of considering the appropriate damping model in the structure and partly because of the unavailability of complex eigensolution sensitivities. Although, there have been considerable research efforts toward damping models, sensitivity of complex eigenvalues and eigenvectors with respect to system parameters appear to have received very little attention in the existing literature.

In this paper we determine the rates of change of complex natural frequencies and mode shapes with respect to some set of design variables in nonproportionally damped discrete linear systems. The assumption is made that the system does not posses repeated eigenvalues. In Sec. II, we briefly discuss the requisite mathematical background on linear multiple-degree-of-freedom discrete systems needed for further derivations. Sensitivity of complex eigenvalues is derived in Sec. III in terms of complex modes, natural frequencies, and changes in the system property matrices. The approach taken here avoids the use of state-space formulation. In Sec. IV, sensitivity of complex eigenvectors is derived. The derivation method uses state-space representation of equations of motion for intermediate calculations and then relates the eigenvector sensitivities to the complex eigenvectors of the second-order system and to the changes in the system property matrices. In Sec. V, a two-degree-of-freedom system that shows the curve-veering phenomenon has been considered to illustrate the application of the expression for rates of change of complex eigenvalues and eigenvectors. The results are carefully

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analyzed and compared with presently available sensitivity expressions of undamped real modes.

II. Background of Analytical Methods

The equations of motion for free vibration of a linear damped discrete system with N degrees of freedom can be written as

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0, \qquad t \geq 0 \tag{1}$$

where M, C, and $K \in \mathbb{R}^{N\times N}$ are mass, damping, and stiffness matrices, $u(t) \in \mathbb{R}^N$ is the vector of the generalized coordinates, and $t \in \mathbb{R}^+$ denotes time. We seek a harmonic solution of the form $u(t) = u\exp[st]$, where $s = i\omega$ with $i = \sqrt{-1}$ and ω denotes frequency. Substitution of $u(t)$ in Eq. (1) results in

$$s^2Mu + sCu + Ku = 0 \tag{2}$$

This equation is satisfied by the ith latent root s_i and ith latent vector u_i of the λ-matrix problem,[10] so that

$$s_i^2Mu_i + s_iCu_i + Ku_i = 0, \qquad \forall i = 1,\ldots,N \tag{3}$$

In the context of structural dynamics, the u_i are called mode shapes, and the natural frequencies λ_i are defined by $s_i = i\lambda_i$. Unless system (1) is proportionally damped, i.e., C is simultaneously diagonalizable with M and K (conditions were derived by Caughey and O'Kelly[11]), in general $\lambda_i \in \mathbb{C}$ and $u_i \in \mathbb{C}^N$. Several authors have proposed methods to obtain complex modes and natural frequencies in N space. Rayleigh[1] considered approximate methods to determine λ_i and u_i by assuming the elements of C are small but otherwise general. Using perturbation analysis, Cronin[12] has given a power series expression of eigenvalues and eigenvectors. Recently Woodhouse[13] has extended Rayleigh's analysis to the case of more general linear damping models described by convolution integrals of the generalized coordinates over the damping kernel functions. A. Bhaskar (private communication, Cambridge, England, United Kingdom, April 1998) developed a procedure to exactly obtain λ_i and u_i by an iterative method. All of these methods calculate the complex modes and frequencies with a varying degree of accuracy depending on various factors: for example, amount of damping, separation between the modes, and number of terms retained in perturbation expansion, etc.

However, complex modes and frequencies can be exactly obtained by the state-space (first-order) formalisms. Transforming Eq. (1) into state-space form, we obtain

$$\dot{\mathfrak{z}}(t) = A\mathfrak{z}(t) \tag{4}$$

where $A \in \mathbb{R}^{2N\times 2N}$ is the system matrix and $\mathfrak{z}(t) \in \mathbb{R}^{2N}$ response vector in the state space given by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \qquad \mathfrak{z}(t) = \begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix} \tag{5}$$

In the preceding equation $0 \in \mathbb{R}^{N\times N}$ is the null matrix, and $I \in \mathbb{R}^{N\times N}$ is the identity matrix. The eigenvalue problem associated with the preceding equation is now in the term of an asymmetric matrix and can be expressed as

$$Az_i = s_iz_i, \qquad \forall i = 1,\ldots,2N \tag{6}$$

where s_i is the ith eigenvalue and $z_i \in \mathbb{C}^{2N}$ is the ith right eigenvector, which is related to the eigenvector of the second-order system as

$$z_i = \begin{Bmatrix} u_i \\ s_iu_i \end{Bmatrix} \tag{7}$$

The left eigenvector $y_i \in \mathbb{C}^{2N}$ associated with s_i is defined by the equation

$$y_i^TA = s_iy_i^T \tag{8}$$

where $(\bullet)^T$ denotes matrix transpose. For distinct eigenvalues one can easily see that the right and left eigenvectors satisfy an orthogonality relationship, that is,

$$y_j^Tz_i = 0, \qquad \forall j \neq i \tag{9}$$

and we may also normalize the eigenvectors so that

$$y_i^Tz_i = 1 \tag{10}$$

The preceding two equations imply that the dynamic system defined by Eq. (4) possess a set of biorthonormal eigenvectors. As a special case, when all eigenvalues are distinct this set forms a complete set. Henceforth in our discussion the assumption will be made that all of the system eigenvalues are distinct.

III. Rates of Change of Eigenvalues

Suppose the structural system defined in Eq. (1) can be described by a set of m parameters (design variables), $g = \{g_1, g_2, \ldots, g_m\}^T \in \mathbb{R}^m$, so that the mass, damping, and stiffness matrices become functions of g, that is, M, C, and $K: g \to \mathbb{R}^{N\times N}$. Assume further that the design variables undergo a small change of the form $\Delta g = \{\Delta g_1, \Delta g_2, \ldots, \Delta g_m\}^T \in \mathbb{R}^m$. For this small change neglecting higher-order terms in the Taylor series, the ith complex eigenvalue can be expressed as

$$\lambda_i^{(c)} \approx \lambda_i + \Delta g^T\nabla\lambda_i \tag{11}$$

where $\lambda_i^{(c)} \in \mathbb{C}$ denotes the changed complex eigenvalue and $\nabla\lambda_i = \{\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,m}\}^T \in \mathbb{C}^m$. Here $\lambda_{i,j} = \partial\lambda_i/\partial g_j$ is the rate of change of the ith eigenvalue with respect to g_j, which is to be found. Recently Bhaskar[14] has derived an expression for $\lambda_{i,j}$ by converting Eq. (3) to the state-space form where the eigenvalue problem takes the Duncan form. Here we try to derive an expression of $\lambda_{i,j}$ without going into the state space.

For ith set substituting $s_i = i\lambda_i$, Eq. (3) can be rewritten as

$$F_iu_i = 0 \tag{12}$$

where the regular matrix pencil

$$F_i \equiv F(\lambda_i, g) = -\lambda_i^2M + i\lambda_iC + K \tag{13}$$

Premultiplication of Eq. (12) by u_i^T yields

$$u_i^TF_iu_i = 0 \tag{14}$$

Differentiating the preceding equation with respect to g_j, one obtains

$$u_{i,j}^TF_iu_i + u_i^TF_{i,j}u_i + u_i^TF_iu_{i,j} = 0 \tag{15}$$

where $F_{i,j}$ stands for $\partial F_i/\partial g_j$ and can be obtained by differentiating Eq. (13) as

$$F_{i,j} = \left[\lambda_{i,j}(iC - 2\lambda_iM) - \lambda_i^2M_{,j} + i\lambda_iC_{,j} + K_{,j}\right] \tag{16}$$

Now taking the transpose of Eq. (12) and using the symmetry property of F_i, one can see that the first and third terms of Eq. (15) are zero. Therefore we have

$$u_i^TF_{i,j}u_i = 0 \tag{17}$$

Substituting $F_{i,j}$ from Eq. (16) into the preceding equation, one writes

$$-\lambda_{i,j}u_i^T(iC - 2\lambda_iM)u_i = u_i^T\left(-\lambda_i^2M_{,j} + i\lambda_iC_{,j} + K_{,j}\right)u_i \tag{18}$$

and again we note that the scalar term

$$u_i^T(iC - 2\lambda_iM)u_i = -(1/\lambda_i)\left[u_i^TF_iu_i - u_i^T\left(\lambda_i^2M + K\right)u_i\right] \tag{19}$$

Finally, after using Eq. (14) and combining the preceding two equations, we can have

$$\lambda_{i,j} = \lambda_i\frac{u_i^T\left(K_{,j} - \lambda_i^2M_{,j} + i\lambda_iC_{,j}\right)u_i}{u_i^T\left(\lambda_i^2M + K\right)u_i} \tag{20}$$

which is the rate of change of the ith complex eigenvalue. For the undamped case when $C = 0$, $\lambda_i \to \omega_i$ and $u_i \to x_i$ (ω_i and x_i are undamped natural frequencies and modes satisfying $Kx_i = \omega_i^2Mx_i$), with usual mass normalization the denominator $\to 2\omega_i^2$, and we obtain

$$2\omega_i\omega_{i,j} = \left(\omega_i^2\right)_{,j} = x_i^T\left[K_{,j} - \omega_i^2M_{,j}\right]x_i \tag{21}$$

This is exactly the well-known relationship derived by Fox and Kapoor[2] for the undamped eigenvalue problem. Thus, Eq. (20) can be viewed as a generalization of the familiar expression of rates of change of undamped eigenvalues to the damped case. The following observations may be noted from this result:

1) The derivative of a given eigenvalue requires the knowledge of only the corresponding eigenvalue and eigenvector under consideration, and thus a complete solution of the eigenproblem, or from the experimental point of view, eigensolution determination for all of the modes is not required.

2) Changes in mass and/or stiffness introduce more change in the real part of the eigenvalues, whereas changes in the damping introduce more change in the imaginary part.

Because $\lambda_{i,j}$ is complex in Eq. (20), it can be effectively used to determine the rates of change of Q factors with respect to the system parameters. For small damping the Q factor for the ith mode is expressed as $Q_i = \Re(\lambda_i)/2\Im(\lambda_i)$, with $\Re(\bullet)$ and $\Im(\bullet)$ denoting real and imaginary parts, respectively. Consequently the rate of change can be evaluated from

$$Q_{i,j} = \frac{1}{2}\left[\frac{\Re(\lambda_{i,j})\Im(\lambda_i) - \Re(\lambda_i)\Im(\lambda_{i,j})}{\Im(\lambda_i)^2}\right] \tag{22}$$

This expression may turn out to be useful because we often directly measure the Q factors from experiment.

IV. Rates of Change of Eigenvectors

For a small change in the design variables $\Delta g \in \mathbb{R}^m$, the ith complex eigenvector can be expressed as

$$u_i^{(c)} \approx u_i + [\nabla u_i]\Delta g \tag{23}$$

where $u_i^{(c)} \in \mathbb{C}^N$ denotes the changed complex eigenvector and $[\nabla u_i] = [u_{i,1}, u_{i,2}, \ldots, u_{i,m}]$, $\in \mathbb{C}^{N\times m}$, with $u_{i,j} = \partial u_i/\partial g_j \in \mathbb{C}^N$ is the ith complex modal sensitivity matrix. Because u_i is the first N rows of z_i [see Eq. (7)], we first try to derive $z_{i,j}$ and subsequently obtain $u_{i,j}$ using their relationship.

Differentiating Eq. (6) with respect to g_j, one obtains

$$(A - s_i)z_{i,j} = -(A_{,j} - s_{i,j})z_i \tag{24}$$

Because the assumption has been made that A has distinct eigenvalues, the right eigenvectors z_i form a complete set of vectors. Therefore we can expand $z_{i,j}$ as

$$z_{i,j} = \sum_{l=1}^{2N} a_{ijl} z_l \tag{25}$$

where a_{ijl}, $\forall l = 1, \ldots, 2N$ are sets of complex constants to be determined. Substituting $z_{i,j}$ in Eq. (24) and premultiplying by the left eigenvector y_k^T, one obtains the scalar equation

$$\sum_{l=1}^{2N} \left(y_k^T A z_l - s_i y_k^T z_l\right) a_{ijl} = -y_k^T A_{,j} z_i + s_{i,j} y_k^T z_i \tag{26}$$

Using the orthogonality relationship of left and right eigenvectors from the preceding equation, we obtain

$$a_{ijk} = \frac{y_k^T A_{,j} z_i}{s_i - s_k}, \qquad \forall k = 1, \ldots, 2N;\ \neq i \tag{27}$$

The a_{ijk} as expressed in Eq. (27) is not very useful because it is in terms of the left and right eigenvectors of the first-order system. To obtain a relationship with the eigenvectors of the second-order system, we assume

$$y_i = \begin{Bmatrix} y_{1_i} \\ y_{2_i} \end{Bmatrix} \tag{28}$$

where $y_{1_i}, y_{2_i} \in \mathbb{C}^N$. Substituting y_i in Eq. (8) and taking transpose, one obtains

$$s_i y_{1_i} = -KM^{-1} y_{2_i}$$

$$s_i y_{2_i} = y_{1_i} - CM^{-1} y_{2_i} \qquad \text{or} \qquad y_{1_i} = \left(s_i I + CM^{-1}\right) y_{2_i} \tag{29}$$

Elimination of y_{1_i} from the preceding two equations yields

$$s_i\left(s_i y_{2_i} + CM^{-1} y_{2_i}\right) = -KM^{-1} y_{2_i} \quad \text{or}$$

$$\left(s_i^2 M + s_i C + K\right)\left(M^{-1} y_{2_i}\right) = 0 \tag{30}$$

By comparison of this equation with Eq. (3), one can see that the vector $M^{-1} y_{2_i}$ is parallel to u_i; that is, there exists a nonzero $\beta_i \in \mathbb{C}$ such that

$$M^{-1} y_{2_i} = \beta_i u_i \qquad \text{or} \qquad y_{2_i} = \beta_i M u_i \tag{31}$$

Now substituting y_{1_i}, y_{2_i} and using the definition of z_i from Eq. (7) into the normalization condition (10), the scalar constant β_i can be obtained as

$$\beta_i = \frac{1}{u_i^T (2 s_i M + C) u_i} \tag{32}$$

Using y_{2_i} from Eq. (31) in the second equation of (29), we obtain

$$y_i = \beta_i P_i z_i, \quad \text{where} \quad P_i = \begin{bmatrix} s_i M + C & 0 \\ 0 & M/s_i \end{bmatrix} \in \mathbb{C}^{2N\times 2N} \tag{33}$$

The preceding equation along with the definition of z_i in Eq. (7) completely relates the left and right eigenvectors of the first-order system to the eigenvectors of the second-order system.

The derivative of the system matrix A can be expressed as

$$A_{,j} = \begin{bmatrix} 0 & 0 \\ (M^{-1}K)_{,j} & (M^{-1}C)_{,j} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -M^{-2}M_{,j}K + M^{-1}K_{,j} & -M^{-2}M_{,j}C + M^{-1}C_{,j} \end{bmatrix} \tag{34}$$

from which, after some simplifications, the numerator of the right-hand side of Eq. (27) can be obtained as

$$y_k^T A_{,j} z_i = -\beta_k u_k^T\left[-M^{-1}M_{,j}(K + s_i C) + C_{,j} + K_{,j}\right]u_i \tag{35}$$

Because $I = MM^{-1}$, $I_{,j} = M_{,j}M^{-1} + M[-M^{-2}M_{,j}] = 0$, or $M_{,j}M^{-1} = M^{-1}M_{,j}$, that is, M^{-1} and $M_{,j}$ commute in product. Using this property and also from Eq. (3) noting that $s_i^2 u_i = -M^{-1}[s_i C + K]u_i$, we finally obtain

$$a_{ijk} = -\beta_k \frac{u_k\left(s_i^2 M_{,j} + s_i C_{,j} + K_{,j}\right)u_i}{s_i - s_k}, \qquad \forall k = 1, \ldots, 2N;\ \neq i \tag{36}$$

This equation relates the a_{ijk} with the complex modes of the second-order system.

To obtain a_{iji}, we begin with the differentiation of the normalization condition (10) with respect to g_j and obtain the relationship

$$y_{i,j}^T z_i + y_i^T z_{i,j} = 0 \tag{37}$$

Substitution of y_i from Eq. (33) further leads to

$$\beta_i\left(z_{i,j}^T P_i^T z_i + z_i^T P_{i,j}^T z_i + z_i^T P_i^T z_{i,j}\right) = 0 \tag{38}$$

where $P_{i,j}$ can be derived from Eq. (33) as

$$P_{i,j} = \begin{bmatrix} s_{i,j}M + s_i M_{,j} + C_{,j} & 0 \\ 0 & -\left(M/s_i^2\right)s_{i,j} + M_{,j}/s_i \end{bmatrix} \tag{39}$$

Because P_i is a symmetric matrix, Eq. (38) can be rearranged as

$$2\left(\beta_i z_i^T P_i\right) z_{i,j} = -\beta_i z_i^T P_{i,j} z_i \tag{40}$$

Note that the term within the bracket is y_i^T [see Eq. (33)]. Using the assumed expansion of $z_{i,j}$ from Eq. (27), this equation reads

$$2 y_i^T \sum_{l=1}^{2N} a_{ijl} z_l = -\beta_i z_i^T P_{i,j} z_i \tag{41}$$

The left-hand side of the preceding equation can be further simplified:

$$\mathbf{z}_i^T \mathbf{P}_{i,j} \mathbf{z}_i = \mathbf{u}_i^T (s_{i,j}\mathbf{M} + s_i \mathbf{M}_{,j} + \mathbf{C}_{,j})\mathbf{u}_i + \mathbf{u}_i^T s_i\big[-\big(\mathbf{M}/s_i^2\big)s_{i,j} + \mathbf{M}_{,j}/s_i\big]s_i\mathbf{u}_i = \mathbf{u}_i^T(2s_i\mathbf{M}_{,j} + \mathbf{C}_{,j})\mathbf{u}_i \tag{42}$$

Finally using the orthogonality property of left and right eigenvectors, from Eq. (41) we obtain

$$a_{iji} = -\frac{1}{2}\frac{\mathbf{u}_i^T(2s_i\mathbf{M}_{,j} + \mathbf{C}_{,j})\mathbf{u}_i}{\mathbf{u}_i^T(2s_i\mathbf{M} + \mathbf{C})\mathbf{u}_i} \tag{43}$$

In the preceding equation a_{iji} is expressed in terms of the complex modes of the second-order system. Now recalling the definition of z_i in Eq. (7), from the first N rows of Eq. (25) one can write

$$\mathbf{u}_{i,j} = a_{iji}\mathbf{u}_i + \sum_{k\neq i}^{2N} a_{ijk}\mathbf{u}_k = -\frac{1}{2}\frac{\mathbf{u}_i^T(2s_i\mathbf{M}_{,j} + \mathbf{C}_{,j})\mathbf{u}_i}{\mathbf{u}_i^T(2s_i\mathbf{M} + \mathbf{C})\mathbf{u}_i}\mathbf{u}_i - \sum_{k\neq i}^{2N}\beta_k\frac{\mathbf{u}_k\big(s_i^2\mathbf{M}_{,j} + s_i\mathbf{C}_{,j} + \mathbf{K}_{,j}\big)\mathbf{u}_i}{s_i - s_k}\mathbf{u}_k \tag{44}$$

We know that for any real symmetric system first-order eigenvalues and eigenvectors appear in complex conjugate pairs. Using usual definition of natural frequency, that is, $s_k = \mathrm{i}\lambda_k$ and consequently $s_k^* = -\mathrm{i}\lambda_k^*$, where $(\bullet)^*$ denotes complex conjugate, the preceding equation can be rewritten in a more convenient form as

$$\mathbf{u}_{i,j} = -\frac{1}{2}\frac{\mathbf{u}_i^T(\mathbf{M}_{,j} - \mathrm{i}\mathbf{C}_{,j}/2\lambda_i)\mathbf{u}_i}{\mathbf{u}_i^T(\mathbf{M} - \mathrm{i}\mathbf{C}/2\lambda_i)\mathbf{u}_i}\mathbf{u}_i + \sum_{k\neq i}^{N}\left[\frac{\alpha_k\big(\mathbf{u}_k^T\tilde{\mathbf{F}}_{i,j}\mathbf{u}_i\big)\mathbf{u}_k}{\lambda_i - \lambda_k} - \frac{\alpha_k^*\big(\mathbf{u}_k^{*T}\tilde{\mathbf{F}}_{i,j}^*\mathbf{u}_i^*\big)\mathbf{u}_k^*}{\lambda_i + \lambda_k^*}\right] \tag{45}$$

where

$$\tilde{\mathbf{F}}_{i,j} = \big(\mathbf{K}_{,j} - \lambda_i^2\mathbf{M}_{,j} + \mathrm{i}\lambda_i\mathbf{C}_{,j}\big)$$

$$\alpha_k = \mathrm{i}\beta_k = \frac{1}{\mathbf{u}_k^T(2\lambda_k\mathbf{M} - \mathrm{i}\mathbf{C})\mathbf{u}_k}$$

This result is a generalization of the known expression of rates of change of real undamped eigenvectors to complex eigenvectors. The following observations can be made from this result:

1) Unlike the eigenvalue derivative the derivative of a given complex eigenvector requires the knowledge of all of the other complex eigenvalues and eigenvectors.

2) The sensitivity depends very much on the modes whose frequency is close to that of the considered mode.

3) Like eigenvalue derivative, changes in mass and/or stiffness introduce more changes in the real part of the eigenvector, whereas changes in damping introduce more changes in the imaginary part.

From Eq. (45) one can easily see that in the undamped limit $\mathbf{C} \to 0$, and consequently $\lambda_k, \lambda_k^* \to \omega_k$; $\mathbf{u}_k, \mathbf{u}_k^* \to \mathbf{x}_k$; $\tilde{\mathbf{F}}_{i,j}, \tilde{\mathbf{F}}_{i,j}^* \to (\mathbf{K}_{,j} - \omega_i^2\mathbf{M}_{,j})$ and also with usual mass normalization of the undamped modes $\alpha_k, \alpha_k^* \to 1/2\omega_k$ reduces the preceding equation exactly to the corresponding well-known expression derived by Fox and Kapoor[2] for derivatives of undamped modes.

V. Example: Two-Degree-of-Freedom System

A. Rates of Change of Eigenvalues

A simple two-degree-of-freedom system has been considered to illustrate a possible use of the expressions developed so far. Figure 1 shows the example taken together with the numerical values. When eigenvalues are plotted vs a system parameter, they create a family of root loci. When two loci approach together, they may cross or rapidly diverge. The latter case is called curve veering. During veering, rapid changes take place in the eigensolutions, as Leissa[15] pointed out: "... the (eigenfunctions) must undergo violent change—figuratively speaking, a dragonfly one instant, a butterfly the next, and something indescribable in between." Thus, this is an interesting problem for applying the general results derived in this paper.

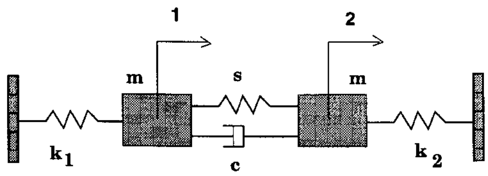

Fig. 1 Two-degree-of-freedom system shows veering; $m = 1$ kg, $k_1 = 1000$ N/m, and $c = 4.0$ Ns/m.

Figure 2 shows the imaginary part [normalized by dividing with $\surd(k_1/m)$] of the rate of change of first natural frequency with respect to the damping parameter c over a parameter variation of k_2 and s. This plot was obtained by direct programming of Eq. (20) in MATLAB™. The imaginary part has been chosen to be plotted here because a change in damping is expected to contribute a significant change in the imaginary part. The sharp rise of the rate in the low-value region of k_2 and s could be intuitively guessed because there the damper becomes the only connecting element between the two masses and so any change made there is expected to have a strong effect. As we move near to the veering range ($k_2 \approx k_1$ and $s \approx 0$), the story becomes quite different. In the first mode the two masses move in the same direction; in fact in the limit the motion approaches a rigid body mode. Here, the change no longer remains sensitive to the changes in connecting the element (i.e., only the damper because $s \approx 0$) as hardly any force transmission takes place between the two masses. For this reason we expect a sharp fall in the rate of change as can be noticed along the $s \approx 0$ region of the figure. For the region when s is large, we also observe a lower value of rate of change, but the reason there is different. The stiffness element s shares most of the force being transmitted between the two masses and hence does not depend much on the change of the value of the damper. A similar plot has been shown in Fig. 3 for the second natural frequency. Unlike the preceding case, here the rate of change increases in the veering range. For the second mode the masses move in the opposite direction, and in the veering range the difference between them becomes maximal. Because $s \approx 0$, only the damper is being stretched, and as a result of this, a small change there produces a large effect. Thus, the use of Eq. (20) can provide good physical insight into the problem and can effectively be used in modal updating, damage detection, and for design purposes by taking the damping matrix together with the mass and stiffness matrices improving the current practice of using the mass and stiffness matrices only.

B. Rates of Change of Eigenvectors

Rates of change of eigenvectors for the problem shown in Fig. 1 can directly be obtained from Eq. (45). Here we have focused our attention to calculate the rates of change of eigenvectors with respect to the parameter k_2. Figure 4 shows the real part of rates of change of the first eigenvector normalized by its $\mathcal{L}^2$ norm (that is, $\Re\{\mathrm{d}\mathbf{u}_1/\mathrm{d}k_2\}/\|\mathbf{u}\|$) plotted over a variation of k_2/k_1 from 0 to 3 for both the coordinates. The value of the spring constant for the connecting spring is kept fixed at $s = 100$ N/m. The real part of the sensitivity of complex eigenvectors has been chosen mainly for two reasons: 1) any change in stiffness is expected to have made more changes in the real part and 2) to compare it with the corresponding changes of the real undamped modes. The derivative of the first eigenvector (normalized by its $\mathcal{L}^2$ norm) with respect to k_2 corresponding to the undamped system (i.e., removing the damper) is also shown in Fig. 4 (see the figure legend for details). This is calculated from the expression derived by Fox and Kapoor.[2] Similar plots for the second eigenvector are shown in Fig. 5. Both of these figures reveal a common feature: around the veering range, i.e., $0.5 < k_2/k_1 < 1.5$, the damped and the undamped sensitivities show considerable differences, whereas outside this region they almost traces each other. A physical explanation of this phenomenon can be given. For the problem considered here the damper acts as an additional connecting element between the two masses together with the spring s. As a result the damper prevents the system from being

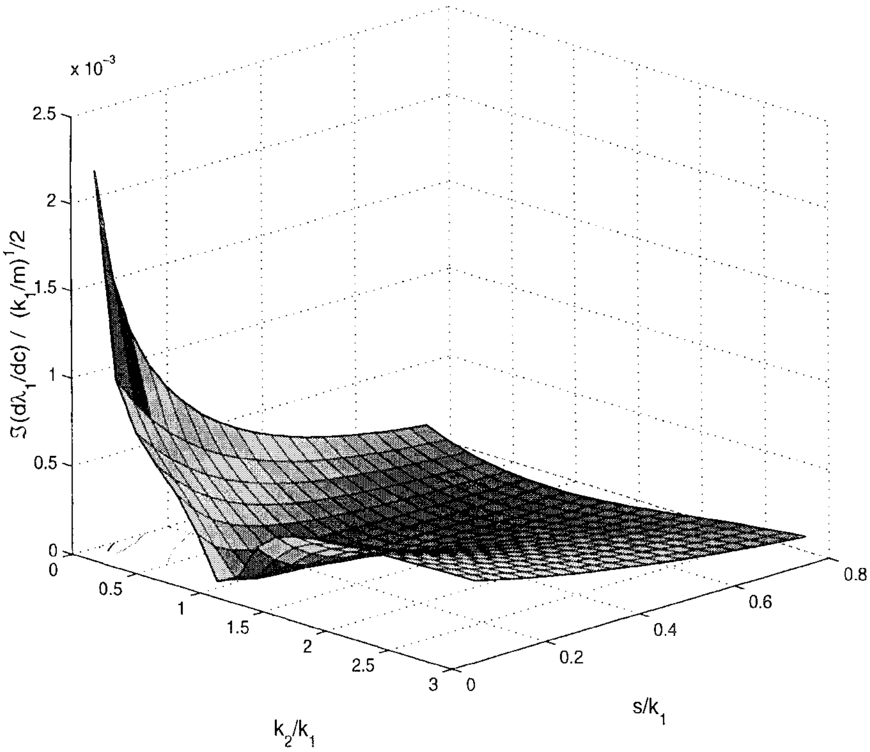

Fig. 2 Imaginary part of rate of change of the first natural frequency λ_1 with respect to the damping parameter c.

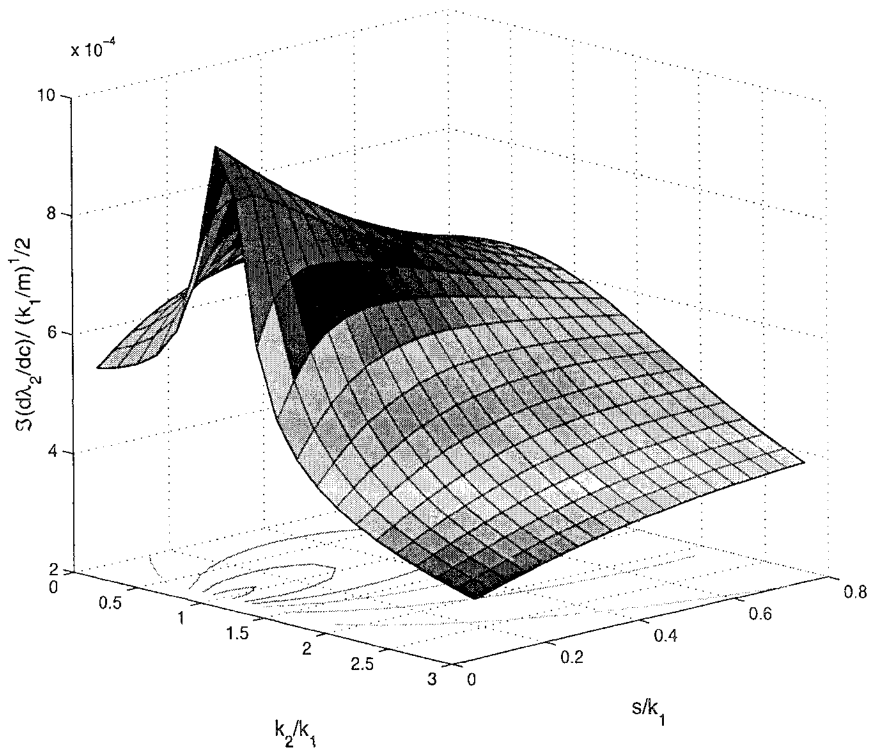

Fig. 3 Imaginary part of rate of change of the second natural frequency λ_2 with respect to the damping parameter c.

closed to show a strong veering effect (i.e., when $k_2 = k_1$ and the force transmission between the masses is close to zero) and thus reduces the sensitivity of both the modes. However, for the first mode both masses move in the same direction, and the damper has less effect compared with the second mode where the masses move in opposite directions and have much greater effect on the sensitivities.

To analyze the results from a quantitative point of view at this point, looking at the variation of the modal Q factors shown in Fig. 6 is interesting. For the first mode Q factor is quite high (in the order of $\approx 10^3$, i.e., quite less damping) near the veering range, but still the sensitivities of the undamped mode and that of the real part of the complex mode for both coordinates are quite different. Again, away from the veering range $k_2/k_1 > 2$, the Q factor is low, but the sensitivities of the undamped mode and that of the real part of the complex mode are quite similar. This is opposite to what we normally expect, as the common belief is that, when the Q factors are high, that is, modal dampings are less, the undamped modes and the real part of complex modes should behave similarly and vice

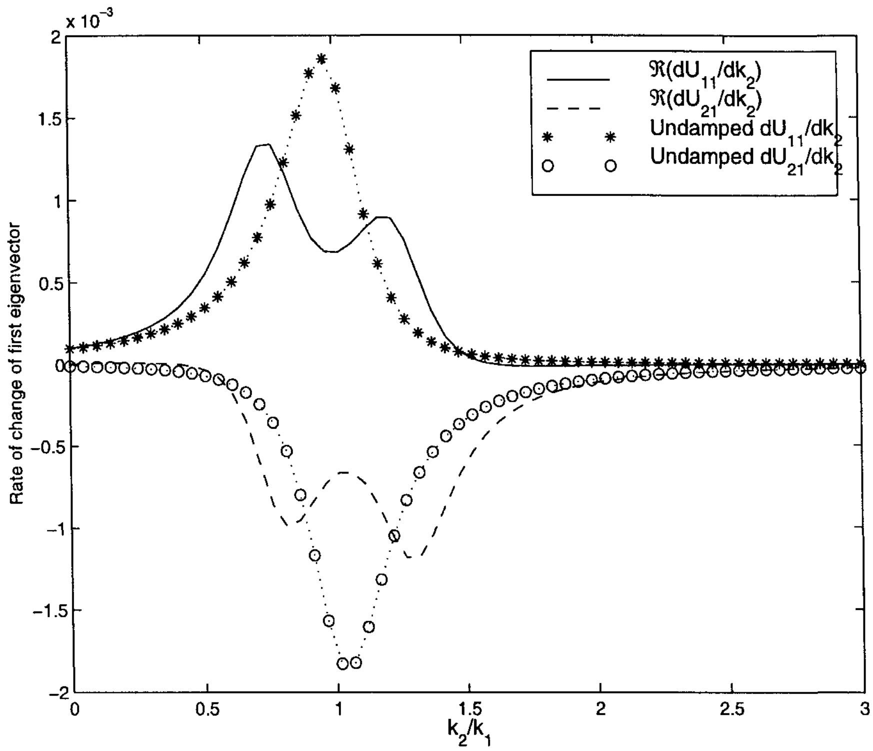

Fig. 4 Real part of rate of change of the first eigenvector with respect to the stiffness parameter k_2.

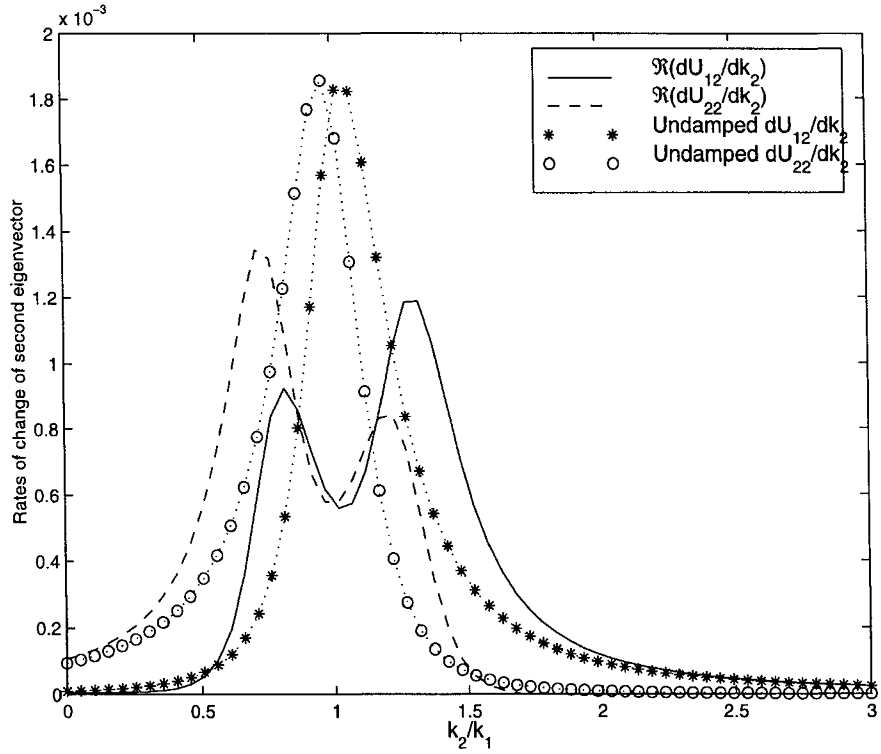

Fig. 5 Real part of rate of change of the second eigenvector with respect to the stiffness parameter k_2.

versa. For the second mode the Q factor does not change very much because of a variation of k_2 except that it becomes a bit lower in the vicinity of the veering range. But the difference between the sensitivities of the undamped mode and that of the real part of the complex mode for both coordinates changes much more significantly than the Q factor. For example $Q_2 \approx 9$ for $k_2/k_1 = 1$ and $Q_2 \approx 11$ for $k_2/k_1 = 2$, but the sensitivity of the undamped mode and that of the real part of the complex mode is much different when $k_2/k_1 = 1$ and quite similar when $k_2/k_1 = 2$. This demonstrates that, even when the Q factors are similar, the sensitivity of the undamped modes and that of the real part of the complex modes can be significantly different. Thus, use of the expression for derivatives of undamped mode shapes can lead to a significant error even when the damping is very low, and the expressions derived in this paper should be used for any kind of study involving such a sensitivity analysis.

Because the expression in Eqs. (20) and (45) has been derived exactly, the numerical results obtained here are also exact within the precision of the arithmetic used for the calculations. The only

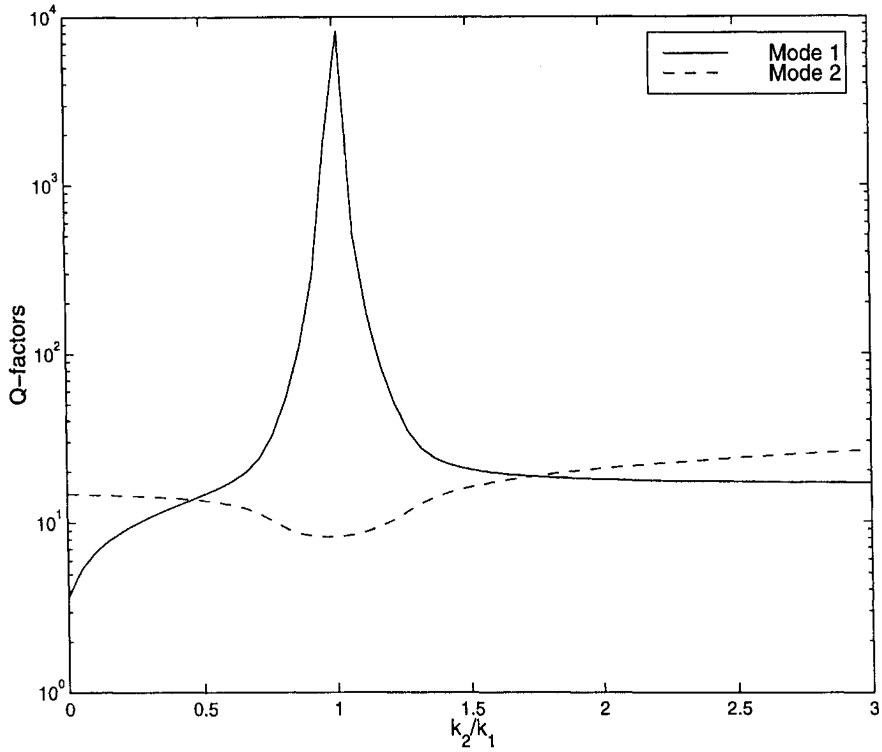

Fig. 6 Q factors for both the modes.

instance for arriving at an approximate result is when approximate complex frequencies and modes are used in the analysis. However, for this example the verification was made that the use of approximate methods to obtain complex eigensolutions in N space reported in the literature[12,13] and by Bhaskar (private communication, Cambridge, April 1998) and the exact ones obtained from the state-space method produce negligible discrepancy. Because in most engineering applications we normally do not encounter very high value of damping, one can use approximate methods to obtain eigensolusions in N space in conjunction with the sensitivity expressions derived here; this will allow the analyst to study the rates of change of eigenvalues and eigenvectors of nonclassically damped systems in a way similar to those of undamped systems.

VI. Conclusion

Rates of change of eigenvalues and eigenvectors of linear damped discrete systems with respect to the system parameters have been derived. In the presence of general nonproportional viscous damping, the eigenvalues and eigenvectors of the system become complex. The results are presented in terms of changes in mass, damping, stiffness matrices, and complex eigensolutions of the second-order system so that the state-space representation of equations of motion can be avoided. The expressions derived hereby generalize earlier results on derivatives of eigenvalues and eigenvectors of undamped systems to the damped systems. Through an example problem the use of the expression for derivative of undamped modes can give rise to erroneous results even when the modal damping is quite low. So for a nonclassically damped system the expressions for rates of change of eigenvalues and eigenvectors developed in this paper should be used. These complex eigensolution derivatives can be useful in various application areas, for example, finite element model updating, damage detection, design optimization, and system stochasticity analysis relaxing the present restriction to use the real undamped modes only.

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References

[1]Rayleigh, Lord, *Theory of Sound*, 2nd ed., Dover, New York, 1945, Sec. 102.

[2]Fox, R. L., and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[3]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[4]Adelman, H. M., and Haftka, R. T., "Sensitivity Analysis of Discrete Structural System," *AIAA Journal*, Vol. 24, No. 5, 1986, pp. 823–832.

[5]Caughey, T. K., "Classical Normal Modes in Damped Linear Dynamic System," *Journal of Applied Mechanics*, Vol. 27, June 1960, pp. 269–271.

[6]Newland, D. E., *Mechanical Vibration Analysis and Computation*, Longman, Harlow, and Wiley, New York, 1989, pp. 145–178.

[7]Park, I. W., Kim, J. S., and Ma, F., "Characteristics of Modal Coupling in Non-Classically Damped Systems Under Harmonic Excitation," *Journal of Applied Mechanics*, Vol. 61, March 1994, pp. 77–83.

[8]Gawronski, W., and Sawicki, J. T., "Response Errors of Non-Proportionally Lightly Damped Structures," *Journal of Sound and Vibration*, Vol. 200, No. 4, 1997, pp. 543–550.

[9]Sestieri, A., and Ibrahim, S. R., "Analysis of Errors and Approximations in the Use of Modal Coordinates," *Journal of Sound and Vibration*, Vol. 177, No. 2, 1994, pp. 145–157.

[10]Lancaster, P., *Lambda-Matrices and Vibrating System*, Pergamon, London, 1966.

[11]Caughey, T. K., and O'Kelly, M. E. J., "Classical Normal Modes in Damped Linear Dynamic System," *Journal of Applied Mechanics*, Vol. 32, Sept. 1965, pp. 583–588.

[12]Cronin, D. L., "Eigenvalue and Eigenvector Determination for Non-Classically Damped Dynamic Systems," *Computer and Structures*, Vol. 36, No. 1, 1990, pp. 133–138.

[13]Woodhouse, J., "Linear Damping Models for Structural Vibration," *Journal of Sound and Vibration*, Vol. 215, No. 3, 1998, pp. 547–569.

[14]Bhaskar, A., "Rayleigh's Classical Sensitivity Analysis Extended to Damped and Gyroscopic Systems," *Proceeding of IUTAM-IITD International Winter School on Optimum Dynamic Design Using Modal Testing and Structural Dynamic Modification*, edited by T. K. Kundra and B. C. Nakra, Delhi Press, New Delhi, India, 1997, pp. 417–430.

[15]Leissa, A. W., "On a Curve Veering Aberration," *Journal of Applied Mathematics and Physics*, Vol. 25, No. 1, 1974, pp. 99–111.

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